New solution of vertex type tetrahedron equations [1]

V.V. Mangazeev[2], S.M. Sergeev[3] and Yu.G. Stroganov[4]

Institute for High Energy Physics,
Protvino, Moscow Region, Russia

Abstract

In this paper we formulate a new N-state spin integrable model on a three-dimensional lattice with spins interacting round each elementary cube of the lattice. This model can be also reformulated as a vertex type model. Weight functions of the model satisfy tetrahedron equations.

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[2]E-mail: mangazeev@mx.ihep.su

[3]Branch Inst. for Nucl. Phys.,E-mail: sergeev_ms@mx.ihep.su

[4]E-mail: stroganov@mx.ihep.su

1. Introduction

Recently in Ref. [1] Korepanov has constructed a new solution of tetrahedron equations with spin variables lying on the edges of a 3D cubic lattice [2]. It leads to a commuting family of transfer-matrices and possible integrability of this vertex model on the cubic lattice. Corresponding spin variables take $N = 2$ values. There are only 16 nonzero weights $R_{i_1 i_2 i_3}^{j_1 j_2 j_3}$ from 64 possible ones.

Later Hietarinta in Ref. [3] has proposed another vertex solution with 16 nonzero weights. It strongly reminds a solution of Ref. [1], but four weights from sixteen in these two models are different. This model satisfy to a duality B property in a terminology of Ref. [4] and can be reformulated as an interaction-round-cube (IRC) model with weight function $W(a|efg|bcd|h)$ depending on eight surrounding spin variables of an elementary cube of the lattice. Note that Baxter in his paper [5] also has reformulated Zamolodchikov model [6] with spin variables belonging to faces of 3D lattice as an IRC model. We fail to generalize the original Korepanov solution for an arbitrary N.

In this paper we propose a generalization of Hietarinta solution of tetrahedron equations for an arbitrary number N of spin variables. The obtained solution has a simple multiplicative form, depends on four parameters, which have a simple geometric interpretation. Namely, one can choose as parameters of our solution four angles of an arbitrary quadrilateral with two diagonals. Hietarinta solution appears in the case if this quadrilateral can be inscribed in the circle and $N = 2$. Also note that in the case $N = 2$ our solution can be obtained as a particular case of Zamolodchikov model [5, 6].

The paper is organized as follows. In section 2 we recall a formulation of IRC type models , introduce necessary definitions and propose an ansatz for weight functions. In section 3 we show that the tetrahedron equations are reduced for this case to some special identity, which is proved in appendix. Section 4 is devoted to a parameterization of obtained N-state solution. And, at last, in section 5 we discuss briefly results of this paper and its possible generalizations.

2. Formulation of The Model

In this section we recall some definitions following Refs. [7, 8]. Consider a simple cubic lattice $\mathcal{L}$ and at each site of $\mathcal{L}$ place a spin variable taking its values in Z_N, for any integer $N \geq 2$ (elements of Z_N are given by N distinct numbers $0, 1, \ldots, N-1$ considered modulo N). Allow all possible interactions of the spins within each elementary cube. The partition function reads

$$Z = \sum_{spins} \prod_{cubes} W(a|e,f,g|b,c,d|h), \tag{2.1}$$

where $W(a|e,f,g|b,c,d|h)$ is the Boltzmann weight of the spin configuration $a, \ldots, h$.

We need in some notations to formulate our ansatz for the Boltzmann weights. Denote

$$\omega = \exp(2\pi i/N), \qquad \omega^{1/2} = \exp(\pi i/N). \tag{2.2}$$

Taking x, y, z to be complex parameters constrained by the Fermat equation

$$x^N + y^N = z^N \tag{2.3}$$

and l to be an element of Z_N, define

$$w(x,y,z|l) = \prod_{s=1}^{l} \frac{y}{z - x\omega^s}. \tag{2.4}$$

This function has a following property

$$w(x,y,z|l) w(z, \omega^{1/2} y, \omega x| -l) \Phi(l) = 1, \quad l \in Z_N, \tag{2.5}$$

where

$$\Phi(l) = \omega^{l(l+N)/2}. \tag{2.6}$$

The tetrahedron equations can be written as (Eqs. (2.2) of Ref. [5])

$$\begin{aligned} &\sum_d \quad W(a_4|c_2,c_1,c_3|b_1,b_3,b_2|d) W'(c_1|b_2,a_3,b_1|c_4,d,c_6|b_4) \\ &\quad \times \quad W''(b_1|d,c_4,c_3|a_2,b_3,b_4|c_5) W'''(d|b_2,b_4,b_3|c_5,c_2,c_6|a_1) \\ &= \sum_d \quad W'''(b_1|c_1,c_4,c_3|a_2,a_4,a_3|d) W''(c_1|b_2,a_3,a_4|d,c_2,c_6|a_1) \\ &\quad \times \quad W'(a_4|c_2,d,c_3|a_2,b_3,a_1|c_5) W(d|a_1,a_3,a_2|c_4,c_5,c_6|b_4), \end{aligned} \tag{2.7}$$

where W, W', W'' and W''' are some four sets of Boltzmann weights. As has been shown in Refs. [2, 9], this relation ensures commutativity of layer-to-layer transfer matrices constructed from W and W' weights:

$$T(W)T(W') = T(W')T(W). \tag{2.8}$$

We are looking for a solution for (2.7) in the following form:

$$W(a|efg|bcd|h) = \omega^{(h-f)(a-b-e+h)}\times \\ \times\frac{w(p_2,p_{12},p_1|a-b-e+h)w(p_4,p_{34},p_3|-a+c+f-h)}{w(p_6,p_{56},p_5|-b+c-e+f)} \quad , \tag{2.9}$$

where functions w are defined by (2.4) and coordinates p_i, p_{ij} satisfy Fermat constraint (2.3). We imply that the set of weight functions W' depend on parameters p_i', p_{ij}' and etc.

3. Proof of the Tetrahedron Equations

In this section we will show that tetrahedron equations (2.7) for weight functions (2.9) are equivalent to the identity (A.8) provided that parameters of weights satisfy some special algebraic constraints.

Substituting formula (2.9) in (2.7) we obtain

$$\begin{aligned}
&\frac{w(p_2',p_{12}',p_1'|b-a)}{w(p_6''',p_{56}''',p_5'''|c-d)}\frac{w(p_4'',p_{34}'',p_3''|c'-b')}{w(p_6,p_{56},p_5|d'-a')}\times \\
&\sum_{\sigma\in Z_N} w^{-1}(p_6',p_{56}',p_5'|-a-\sigma)w(p_4',p_{34}',p_3'|-b-\sigma)w(p_4''',p_{34}''',p_3'''|c+\sigma) \\
&\times w(p_2''',p_{12}''',p_1'''|-d-\sigma)w(p_4,p_{34},p_3|-b'+\sigma)w(p_2'',p_{12}'',p_1''|-d'+\sigma) \\
&\times w(p_2,p_{12},p_1|b'+d'-a'-\sigma)w^{-1}(p_6'',p_{56}'',p_5''|c'-b'-d'+\sigma)\omega^{\sigma^2+\sigma(c-c')} \\
&= \frac{w(p_4'',p_{34}'',p_3''|c-b)}{w(p_6,p_{56},p_5|d-a)}\frac{w(p_2',p_{12}',p_1'|b'-a')}{w(p_6''',p_{56}''',p_5'''|c'-d')}\times \\
&\sum_{\sigma\in Z_N} w^{-1}(p_6',p_{56}',p_5'|-a'-\sigma)w(p_4',p_{34}',p_3'|-b'-\sigma)w(p_4''',p_{34}''',p_3'''|c'+\sigma) \\
&\times w(p_2''',p_{12}''',p_1'''|-d'-\sigma)w(p_4,p_{34},p_3|-b+\sigma)w(p_2'',p_{12}'',p_1''|-d+\sigma) \\
&\times w(p_2,p_{12},p_1|b+d-a-\sigma)w^{-1}(p_6'',p_{56}'',p_5''|c-b-d+\sigma)\omega^{\sigma^2+\sigma(c'-c)},
\end{aligned} \tag{3.1}$$

where we made a replacement $\sigma \to -\sigma$ in the LHS, $\sigma \to c_1 + c_5 - \sigma$ in the RHS and introduced new spin variables:

$$\begin{aligned}
a &= -a_3 + b_2 + c_4, \qquad & a' &= a_2 - b_3 - c_1 + c_2 - c_5, \\
b &= -a_3 + b_4 + c_1, \qquad & b' &= a_4 - b_3 - c_1, \\
c &= -a_1 + b_4 + c_2, \qquad & c' &= a_4 - b_1 - c_1 + c_4 - c_5, \\
d &= -a_1 + b_2 + c_5, \qquad & d' &= a_2 - b_1 - c_5.
\end{aligned} \tag{3.2}$$

Relation (3.1) looks very similar to identity (A.8) from appendix. But for complete coincidence of spin structure of (3.1) and (A.8) we need to use (2.5). Using (2.5) several times we can easily reduce (3.1) to identity (A.8) with the following identification:

$$\begin{aligned}
\vec{x} &\sim (p_5', \omega^{1/2} p_{56}', \omega p_6'), \qquad & \vec{x}''' &\sim (p_5'', \omega^{1/2} p_{56}'', \omega p_6''), \\
\vec{y} &\sim (p_3', \omega^{1/2} p_{34}', \omega p_4'), \qquad & \vec{y}''' &\sim (p_3, \omega^{1/2} p_{34}, \omega p_4), \\
\vec{z} &\sim (p_4''', p_{34}''', p_3'''), \qquad & \vec{z}''' &\sim (p_2, p_{12}, p_1), \\
\vec{u} &\sim (p_1''', \omega^{1/2} p_{12}''', \omega p_2'''), \qquad & \vec{u}''' &\sim (p_1'', \omega^{1/2} p_{12}'', \omega p_2''), \\
(x_1 y_3, \xi, \omega x_3 y_1) &\sim (p_1', \omega^{1/2} p_{12}', \omega p_2'), \qquad & (y_3 z_1, \xi'', \omega y_1 z_3) &\sim (p_4'', p_{34}'', p_3''), \\
(z_3 u_1, \tilde{\xi}, z_1 u_3) &\sim (p_5''', \omega^{1/2} p_{56}''', \omega p_6'''), \qquad & (x_3 u_1, \tilde{\xi}'', x_1 u_3) &\sim (p_6, p_{56}, p_5).
\end{aligned} \tag{3.3}$$

Since we can choose four vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ and $\vec{u}$ in an arbitrary way, we come to the four-parametric solution of tetrahedron equations (2.7). In the next section we will show that this solution can be parameterized by the angles of a degenerate tetrahedron with all apices lying in one plane.

4. Parameterization of solution

Using formulas (A.5-A.6) from appendix A we can exclude parameters entering in $\vec{x}$, $\vec{y}$, $\vec{z}$ and $\vec{u}$ from relations (3.3). Then we come to the following constraints among parameters p:

$$\frac{p_6}{p_5} = \frac{p_2 p_4}{p_1 p_3}, \quad \frac{p_6'}{p_5'} = \frac{p_2' p_4'}{p_1' p_3'}, \quad \frac{p_6''}{p_5''} = \frac{p_2'' p_4''}{p_1'' p_3''}, \quad \frac{p_6'''}{p_5'''} = \frac{p_2''' p_4'''}{p_1''' p_3'''}, \tag{4.1}$$

$$\frac{p_2'}{p_1'} = \frac{p_2 p_2''}{p_1 p_1''}, \quad \frac{p_4''}{p_3''} = \frac{p_4' p_4'''}{p_3' p_3'''}, \quad \frac{p_4 p_2'''}{p_3 p_1'''} = \frac{p_4' p_2''}{p_3' p_1''}, \tag{4.2}$$

$$\frac{p'_2 p'_{56}}{p'_6 p'_{12}} \frac{p''_6 p''_{12}}{p''_2 p''_{56}} \frac{p'''_2 p'''_{56}}{p'''_6 p'''_{12}} = 1, \frac{p_1 p_{56}}{p_5 p_{12}} \frac{p''_3 p''_{12}}{p''_2 p''_{34}} \frac{p'''_2 p'''_{34}}{p'''_3 p'''_{12}} = 1, \frac{p_5 p_{34}}{p_4 p_{56}} \frac{p'_3 p'_{56}}{p'_5 p'_{34}} \frac{p''_6 p''_{34}}{p''_3 p''_{56}} \frac{p'''_5}{p'''_6} = 1. \quad (4.3)$$

Also we obtain additional constraints on parameters p_i corresponding to formulas (A.6) for ξ'^N and $\tilde{\xi}'^N$ from appendix A. These constraints ensure a periodicity (*modulo* N) of all w functions entering in (2.9) and can be easily obtained as a consequence of (4.1-4.3).

Remind that the function $w(p_j, p_{ij}, p_i|n)$ depends on one complex parameter p_i/p_j and parameter p_{ij} defined by a relation $p_{ij}^N = p_i^N - p_j^N$ can contain an arbitrary multiplier $\omega^{n_{ij}}$, where $n_{ij} \in Z$.

It appears that instead of variables p_i, p_{ij} and p_j it is more convenient to introduce "angle-like" variables a_1, a_2 and a_3 as

$$\begin{aligned} p_2 &= \exp(-ia_1/N), \quad p_1 = \exp(ia_1/N), \quad p_{12} = \{2i\sin(a_1)\}^{1/N}\omega^{n_{12}}, \\ p_4 &= \exp(-ia_2/N), \quad p_3 = \exp(ia_2/N), \quad p_{34} = \{2i\sin(a_2)\}^{1/N}\omega^{n_{34}}, \\ p_6 &= \exp(-ia_3/N), \quad p_5 = \exp(ia_3/N), \quad p_{56} = \{2i\sin(a_3)\}^{1/N}\omega^{n_{56}} \end{aligned} \quad (4.4)$$

and etc.

Then relations (4.1-4.2) are reduced to linear constraints among a_i:

$$\begin{aligned} &a_3 = a_1 + a_2, \quad a'_3 = a'_1 + a'_2, \quad a''_3 = a''_1 + a''_2, \quad a'''_3 = a'''_1 + a'''_2, \\ &a_1 - a'_1 + a''_1 = 0, \quad a'_2 - a''_2 + a'''_2 = 0, \quad a_2 - a'_2 - a''_1 + a'''_1 = 0. \end{aligned} \quad (4.5)$$

Now let us fix a choice of phase multipliers $\omega^{n_{ij}}$. Taking into account formula (2.9) it is easy to see that we have the following dependence of the weight function from n_{ij}:

$$\omega^{n_{12}(a-b-e+h)+n_{34}(-a+c+f-h)+n_{56}(b-c+e-f)}. \quad (4.6)$$

In fact, this multiplier corresponds to a simple gauge transformation of the weight function and hereafter we will omit all multipliers $\omega^{n_{ij}}$.

Then relations (4.3) can be rewritten in the following form:

$$\begin{gathered} \frac{\sin(a'_3)}{\sin(a'_1)} \frac{\sin(a''_1)}{\sin(a''_3)} \frac{\sin(a'''_3)}{\sin(a'''_1)} = 1, \quad \frac{\sin(a_3)}{\sin(a_1)} \frac{\sin(a''_1)}{\sin(a''_2)} \frac{\sin(a'''_2)}{\sin(a'''_1)} = 1, \\ \frac{\sin(a_2)}{\sin(a_3)} \frac{\sin(a'_3)}{\sin(a'_2)} \frac{\sin(a''_2)}{\sin(a''_3)} = 1. \end{gathered} \quad (4.7)$$

Formulas (4.5,4.7) have a nice geometrical interpretation (see Fig. 1)

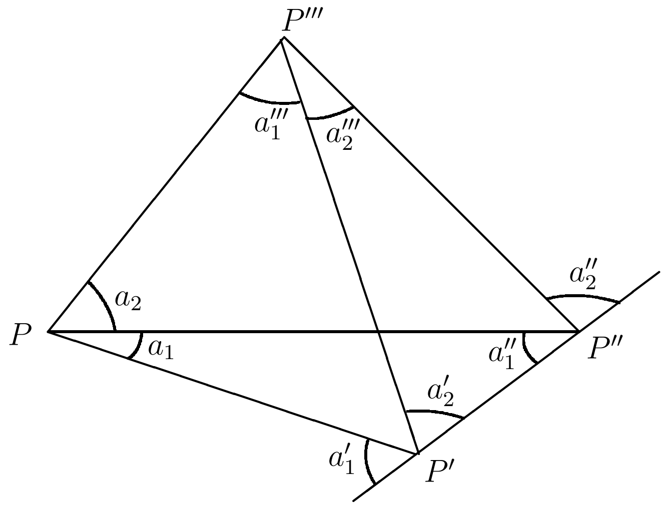

Fig. 1

Then formulas (4.5) are valid in an evident way and formulas (4.7) are the consequences of sine theorem for four triangles formed by two sides and one of the diagonals of the quadrilateral.

5. Discussion

In the previous section we obtained a four-parametric solution of the tetrahedron equations for which each weight function depends on two spectral parameters.

Namely

$$W = W(a_1, a_2), \quad W' = W(a_1', a_2'), \quad W'' = W(a_1'', a_2''), \quad W''' = W(a_1''', a_2'''). \tag{5.1}$$

Weight functions are defined by multiplicative ansatz (2.9) where parameters p_i and p_{ij} can be calculated through angle variables a_1, a_2 from formulas (4.4) (remind that $a_3 = a_1 + a_2$). All set of angle variables a_i, a_i', a_i'' and a_i''' is described by Fig. 1.

It is naturally to ask: is this solution of tetrahedron equations new or not? To clarify this situation we have considered the only known solution of tetra-

hedron equations with $N > 2$ [10] corresponding to the model proposed by Bazhanov and Baxter [7] in the "plane" limit for which all apices of tetrahedron belong to one plane. More precisely, let us consider a parameterization of Ref. [8] and take the limit $\theta_1,\ \theta_2,\ \theta_4,\ \theta_5 \to 0$ and $\theta_3,\ \theta_6 \to \pi$. This limit corresponds explicitly to parameterization (5.1) and to Fig. 1. Formulas for weight functions look as

$$\begin{aligned} W(a|efg|bcd|h) &= (-1)^{a+c+f+h}\omega^{\frac{1}{2}(-a^2-c^2+f^2+h^2+2ag-2bf)} \\ &\Big\{\sum_{\sigma\in Z_N} \frac{w(x_2,\omega^{1/2}x_{23},x_3|b-f+\sigma)}{w(x_4,x_{14},x_1|e-c-\sigma)}\omega^{\sigma(a-c-f+h)}\Big\}_0, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} x_1 &= \exp(ia_2/N)(\sin a_1)^{1/N}, \quad x_4 = \omega^{-1/2}\exp(-ia_1/N)(\sin a_2)^{1/N}, \\ x_2 &= \exp(ia_1/N)(\sin a_2)^{1/N}, \quad x_3 = \omega^{1/2}\exp(-ia_2/N)(\sin a_1)^{1/N}, \\ x_{14} &= (\sin(a_1+a_2))^{1/N}, \quad x_{23} = (\sin(a_1+a_2))^{1/N}, \end{aligned} \tag{5.3}$$

and angles a_i appear as limit values of plane tetrahedron angles. The subscript "0" after the curly brackets implies that the expression in the curly brackets is divided by itself with all exterior spin variables equated to zero. If the signs of σ in w functions entering in (5.2) were equal we would use formula (A14) of appendix A of Ref. [10] and come to a multiplicative solution. It can be done for the case $N = 2$ where all spin variables belong Z_2 and we can change all signs in an appropriate way. Only in this case we obtain from (5.2) our multiplicative solution up to some gauge transformation.

So it seems that multiplicative solution (2.9) should be new for $N \geq 3$. Unfortunately, we did not succeed yet in a generalization of this solution to a five-parametric one which could be described by the angles of the usual tetrahedron.

Note also that solution (2.9) can be naturally rewritten in terms spin variables lying on the edges of the lattice.

Let us define the following spin variables [4]

$$\begin{aligned} i_1 &= -a+c, \quad i_2 = e-f, \quad i_3 = a-b, \\ j_1 &= -f+h, \quad j_2 = -b+c, \quad j_3 = e-h. \end{aligned} \tag{5.4}$$

Then we can rewrite our weight function (2.9) as

$$R^{j_1j_2j_3}_{i_1i_2i_3} = \omega^{j_1(i_3-j_3)}\frac{w(p_2,p_{12},p_1|i_3-j_3)w(p_4,p_{34},p_3|i_1-j_1)}{w(p_6,p_{56},p_5|j_2-i_2)}. \tag{5.5}$$

Note that spin variables i_1, i_2, i_3, j_1, j_2, j_3 should satisfy two constraints

$$j_2 = i_1 + i_3, \quad i_2 = j_1 + j_3. \tag{5.6}$$

Then tetrahedron equations take the vertex form

$$\sum_{\substack{k_1,k_2,k_3,\\ k_4,k_5,k_6}} R^{k_1,k_2,k_3}_{i_1,i_2,i_3} R'^{j_1k_4k_5}_{k_1i_4i_5} R''^{j_2j_4k_6}_{k_2k_4i_6} R'''^{j_3j_5j_6}_{k_3k_5k_6} =$$
$$= \sum_{\substack{k_1,k_2,k_3,\\ k_4,k_5,k_6}} R'''^{k_3,k_5,k_6}_{i_3,i_5,i_6} R''^{k_2k_4j_6}_{i_2i_4k_6} R'^{k_1j_4j_5}_{i_1k_4k_5} R^{j_1j_2j_3}_{k_1k_2k_3} \quad . \tag{5.7}$$

At last let us suppose that the quadrilateral shown in Fig. 1 can be inscribed in the circle. In this case all relations constrained angles a_i, a'_i, a''_i and a'''_i appear to be linear, and we have only three independent angle variables. For $N = 2$ this model coincides with that of Ref. [3] (to be exact, we are to change else $a_j \to ia_j$).

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Appendix

In this Appendix we will prove the identity (3.1), from which it follows the tetrahedron equation for the multiplicative ansatz.

We begin with the following relation:

$$\left\{ \sum_{\sigma\in Z_N} \frac{w(\vec{x}|a+\sigma)w(\vec{z}|c+\sigma)}{w(\vec{y}|b+\sigma)w(\vec{u}|d+\sigma)} \omega^{\sigma n} \right\}_0 = \omega^{-nc} \frac{w(x_1y_3, \xi, \omega x_3y_1|a-b)}{w(z_3u_1, \tilde{\xi}, z_1u_3|d-c)}$$
$$\times \left\{ \sum_{\sigma\in Z_N} \frac{w(\vec{x}'|\sigma)w(\vec{z}'|d-c+n+\sigma)}{w(\vec{y}'|n+\sigma)w(\vec{u}'|a-b+\sigma)} \omega^{\sigma(b-c)} \right\}_0 \tag{A.1}$$

where

$$\vec{x}' = (x_3y_2, \xi, x_2y_3)$$

$$\vec{y}' = (u_3 z_2/\omega, \tilde{\xi}, z_3 u_2)$$
$$\vec{z}' = (u_1 z_2, \tilde{\xi}, u_2 z_1)$$
$$\vec{u}' = (x_1 y_2, \xi, \omega x_2 y_1), \tag{A.2}$$

the vectors $\vec{x}, \vec{y}, \vec{z}, \vec{u}$ are arbitrary, and the auxiliary parameters ξ and $\tilde{\xi}$ obey the relations

$$\xi^N = x_3^N y_1^N - x_1^N y_3^N$$
$$\tilde{\xi}^N = z_1^N u_3^N - z_3^N u_1^N, \tag{A.3}$$

phases of ξ and $\tilde{\xi}$ can be chosen in arbitrary way. The subscript "0" after the curly brackets implies that the expression in the curly brackets is divided by itself with all the exterior spin variables equated to zero.

This relation is equivalent to the relation (B9) from the Appendix B of Ref. [10] and can be interpreted as some symmetry transformation of the Boltzmann weights of the generalization of the Bazhanov – Baxter model, considered in Ref. [11].

Applying this transformation triple, we obtain the following relation:

$$\left\{ \sum_{\sigma \in Z_N} \frac{w(\vec{x}|a+\sigma) w(\vec{z}|c+\sigma)}{w(\vec{y}|b+\sigma) w(\vec{u}|d+\sigma)} \omega^{\sigma n} \right\}_0 =$$
$$\omega^{-n(b+d)+d(a-b+c-d)} \frac{w(x_3 y_2 z_3 u_2, \xi', x_2 y_3 z_2 u_3| -n)}{w(x_1 y_2 z_1 u_2, \tilde{\xi}', \omega x_2 y_1 z_2 u_1 | a-b+c-d-n)}$$
$$\frac{w(x_1 y_3, \xi, \omega x_3 y_1 | a-b)}{w(z_3 u_1, \tilde{\xi}, z_1 u_3 | d-c)} \frac{w(y_3 z_1, \xi'', \omega y_1 z_3 | c-b)}{w(x_3 u_1, \tilde{\xi}'', x_1 u_3 | d-a)}$$
$$\left\{ \sum_{\sigma \in Z_N} \frac{w(\vec{x}'''| -c+\sigma) w(\vec{z}'''| -a+\sigma)}{w(\vec{y}'''| -d+\sigma) w(\vec{u}'''| -b+\sigma)} \omega^{\sigma(-a+b-c+d+n)} \right\}_0 \tag{A.4}$$

where

$$\vec{x}''' = (z_3 \tilde{\xi}', \xi'' \tilde{\xi} x_2, z_1 \xi')$$
$$\vec{y}''' = (u_3 \tilde{\xi}'/\omega, \tilde{\xi}'' \tilde{\xi} y_2, u_1 \xi')$$
$$\vec{z}''' = (x_3 \tilde{\xi}', \tilde{\xi}'' \xi z_2, x_1 \xi')$$
$$\vec{u}''' = (y_3 \tilde{\xi}', \xi'' \xi u_2, \omega y_1 \xi'), \tag{A.5}$$

and

$$\begin{aligned}\xi'^N &= x_2^N y_3^N z_2^N u_3^N - x_3^N y_2^N z_3^N u_2^N\\ \tilde{\xi}'^N &= x_2^N y_1^N z_2^N u_1^N - x_1^N y_2^N z_1^N u_2^N\\ \xi''^N &= y_1^N z_3^N - y_3^N z_1^N\\ \tilde{\xi}''^N &= x_1^N u_3^N - x_3^N u_1^N\end{aligned} \tag{A.6}$$

Note that in the special case when $x_1 y_3 z_1 u_3 = x_3 y_1 z_3 u_1 \omega$ the summands from both parts of (A.4) become the Boltzmann weights for the Bazhanov – Baxter model [7], and (A.4) becomes the well known Star – Star relation.

Consider now the second copy of (A.4) with the same $\vec{x}, \vec{y}, \vec{z}, \vec{u}$, with the same number n and with some spines a', b', c', d' obeying

$$a - b + c - d = a' - b' + c' - d'. \tag{A.7}$$

Multiplying left hand side of one copy of (A.4) by the right hand side of the other copy, we obtain the identity, from which the w – functions containing the spin n are cancel out. Summing the obtained relation by n, we obtain

$$\begin{aligned}&\left\{\sum_{\sigma\in Z_N} \frac{w(\vec{x}|a+\sigma)w(\vec{z}|c+\sigma)}{w(\vec{y}|b+\sigma)w(\vec{u}|d+\sigma)} \frac{w(\vec{x}'''|b'-c'+d'-\sigma)w(\vec{z}'''|b'-a'+d'-\sigma)}{w(\vec{y}'''|b'-\sigma)w(\vec{u}'''|d'-\sigma)}\right.\\ &\left.\times\omega^{(\sigma-b')(a'-b'+c'-d')}\right\}_0 \frac{w(x_1y_3,\xi,\omega x_3y_1|a'-b')}{w(z_3u_1,\tilde{\xi},z_1u_3|d'-c')} \frac{w(y_3z_1,\xi'',\omega y_1z_3|c'-b')}{w(x_3u_1,\tilde{\xi}'',x_1u_3|d'-a')} =\\ &\left\{\sum_{\sigma\in Z_N} \frac{w(\vec{x}|a'+\sigma)w(\vec{z}|c'+\sigma)}{w(\vec{y}|b'+\sigma)w(\vec{u}|d'+\sigma)} \frac{w(\vec{x}'''|b-c+d-\sigma)w(\vec{z}'''|b-a+d-\sigma)}{w(\vec{y}'''|b-\sigma)w(\vec{u}'''|d-\sigma)}\right.\\ &\left.\times\omega^{(\sigma-b)(a-b+c-d)}\right\}_0 \frac{w(x_1y_3,\xi,\omega x_3y_1|a-b)}{w(z_3u_1,\tilde{\xi},z_1u_3|d-c)} \frac{w(y_3z_1,\xi'',\omega y_1z_3|c-b)}{w(x_3u_1,\tilde{\xi}'',x_1u_3|d-a)}\end{aligned} \tag{A.8}$$

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